Physica D 56 (1992) 135–150
North-Holland

PHYSICA D

Remarks on the quasi-classical propagator of area-preserving maps

Georg Junker and Hajo Leschke
Institut für Theoretische Physik, Universität Erlangen-Nürnberg, Staudtstraße 7, W-8520 Erlangen, Germany

Received 9 December 1991
Revised manuscript received 11 February 1992
Accepted 11 February 1992
Communicated by F.H. Busse

The path-integral-like expression for the quantum propagator of discrete-time area-preserving maps is evaluated approximately by neglecting higher than second-order terms in an expansion of the action about the classical paths. In the resulting quasi-classical approximation for the propagator special attention is paid to the recursive nature of its amplitude and the possible appearance of Maslov-like phases. Using a further approximation for the amplitude we arrive at explicit expressions which clearly show the differences between the contributions of stable and unstable classical paths. An estimate for the range of validity for the quasi-classical approximation is also given.

1. Introduction

The properties of quantum systems, which show chaotic behaviour in the classical limit, have now been studied for more than ten years [1–6]. Of particular interest have been the remnants of classical chaos in quantum dynamics. Systems which are intensively worked on are area-preserving maps. They appear in the Poincaré surface of sections of Hamiltonian flows and are easy to simulate on computers. It was found that in the time development of the corresponding quantum model chaos is always suppressed after a long-enough time [1]. Only in the quasi-classical regime quantum dynamics clearly shows signatures of the difference between classically regular and irregular motion [1, 3].

Whereas classical chaos is a long-time phenomenon, the quasi-classical regime is limited to a finite time, because Planck's constant $\hbar$ is non-zero. Therefore, the quasi-classical approximation of a classically chaotic quantum system requires a careful discussion of its range of validity.

A quasi-classical investigation of the propagator for discrete-time area-preserving maps has been performed in the arena of phase space [2]. In this paper we are concerned with the propagator in the configuration space for periodically kicked systems. This kind of propagator was discussed by Tabor [7] only briefly as he was more interested in its trace. However, the propagator does contain more information than its trace. For example, it can be used to calculate quantum-mechanical correlation functions of classically chaotic systems [8].

Our main goal is to present a simple derivation for the propagator within the quasi-classical limit in such a way that it becomes practicable for numerical calculations. In this respect, we do not have a particular system in mind but try to keep the discussion fairly general.

After describing the classical model we approximate the path-integral-like expression for the quantum propagator by expanding the action about the classical path and neglecting deviations of higher than second order. This is the standard path-integral approach to the WKB-approximation in quantum dynamics. We will explicitly perform the integration with particular attention paid to the possible appearance of Maslov-like phases. In section 4 the connection of our result with that of Tabor [7] is established. Within a further approximation for the amplitude of the quasi-classical propagator we find explicit formulas for the contributions of stable and unstable classical paths having different signatures. The validity of this approximation is tested in section 4.2 for a quartic kicking potential. We also present some implications of the quasi-classical approximation for the spectral properties of the quasi-energy operator. In section 5 we estimate higher-order corrections to the quasi-classical approximation. In the conclusion we propose some applications and generalizations of the presented formalism. We also point out the limitations of this approach which has been overlooked in the past. In appendix A we give the exact solution for a quadratic kicking potential. In appendix B we derive for an arbitrary kicking potential the quasi-classical approximation of the trace of the propagator and correct results of ref. [7].

2. The classical model

The discrete-time area-preserving map we are interested in is generated by the time-dependent classical Hamiltonian

$$H(t)=\frac{p^2}{2m}+\sum_{n=-\infty}^{+\infty} V_n(q)\,\delta(n-t/\tau),\quad (p,q)\in\mathbb{R}^2,\quad \tau>0, \tag{2.1}$$

which describes the free motion of a point particle with mass $m>0$ on the full Euclidean line $\mathbb{R}$ subjected to a time-dependent perturbation by delta-kicks of varying strength characterized by a potential $V_n(q)$. The subscript n indicates a possible explicit time dependence of the kicking force. Denoting the constant momentum $p(t)$ for the time between the kick n and $n+1$ by p_n, that is $p_n := p(t)$ for $n\tau < t < (n+1)\tau$, and the position at time $t=n\tau$ by $q_n := q(n\tau)$, the classical equations of motion read in phase space

$$q_n = q_{n-1} + (\tau/m)\,p_{n-1},\qquad p_n = p_{n-1} - \tau V_n'(q_n), \tag{2.2}$$

where $V_n'(q) := \partial V_n(q)/\partial q$. The classical motion for an initial point (p_0, q_0) in phase space $\mathbb{R}^2$ is uniquely determined via (2.2). On the other hand, eliminating the momentum variable gives the following second-order recurrence relation for the position in configuration space:

$$q_{n+1} = 2q_n - q_{n-1} - (\tau^2/m)V_n'(q_n). \tag{2.3}$$

From this equation the position q_N of the particle after N kicks is uniquely determined by the initial pair (q_0, q_1), that is $q_N = q_N(q_1, q_0)$. Or vice versa, for a given initial point q_0 and a given end point q_N the solutions q_1^α of

$$q_N = q_N(q_1^\alpha, q_0) \tag{2.4}$$

uniquely determine the classical "paths" which start in q_0 at $t=0$ and end in q_N at $t=N\tau$. Such a classical path is given by the set of points $\{q_n^\alpha\}_{n=0,1,\dots,N}$ where α enumerates the distinct solutions of (2.4) with $q_0^\alpha = q_0$ and $q_N^\alpha = q_N$.

The action associated with a particular classical path with number α reads

$$S_{\rm cl}^{\alpha}(q_N, q_0) \equiv S^{(N)}(q_N, q_{N-1}^{\alpha}, \ldots, q_1^{\alpha}, q_0) := \sum_{n=1}^{N} S_n(q_n^{\alpha}, q_{n-1}^{\alpha}), \tag{2.5}$$

where the one-kick action is defined by

$$S_n(x, x') := \frac{m}{2\tau}(x - x')^2 - V_n(x)\,\tau. \tag{2.6}$$

Finally, we note that closed orbits characterized by the condition $q_N(q_0, q_1^{\alpha}) = q_0$ are not necessarily periodic orbits as in general $p_N \neq p_0$. The periodic orbits are period-N fixed points in phase space, that is, $(p_n, q_n) = (p_{n+N}, q_{n+N})$, $n \in \mathbb{Z}$.

3. The quasi-classical approximation of the quantum propagator

The propagator or time-evolution operator $\hat{U}^{(N)}$ for the quantum version of the kicked system (2.1) after N kicks is given by the time-ordered product

$$\hat{U}^{(N)} = \hat{U}_N \hat{U}_{N-1} \cdots \hat{U}_2 \hat{U}_1, \tag{3.1}$$

where the one-kick propagator

$$\hat{U}_n := \exp\Big(-\frac{\mathrm{i}}{\hbar} V_n(\hat{q})\,\tau\Big)\exp\Big(-\frac{\mathrm{i}}{\hbar}\frac{\hat{p}^2}{2m}\tau\Big) \tag{3.2}$$

describes free motion for $(n-1)\tau < t < n\tau$ followed by a kick at instant $t = n\tau$. In the above $\hat{q}$ and $\hat{p}$ are the usual position and momentum operators with commutation relation $\hat{q}\hat{p} - \hat{p}\hat{q} = \mathrm{i}\hbar$.

The problem is to find a more explicit expression for the product (3.1). Therefore, we turn to the $\hat{q}$-representation. In this representation the one-kick propagator is given as

$$\langle x|\hat{U}_n|x'\rangle = \sqrt{\frac{m}{2\pi\mathrm{i}\hbar\tau}}\,\exp[(\mathrm{i}/\hbar)S_n(x, x')], \tag{3.3}$$

with $S_n(x, x')$ being the one-kick action (2.6). Throughout this paper the symbol $\sqrt{}$ denotes the principle value of the square root, e.g. $\sqrt{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\pi/4}$. The product (3.1) then takes a Feynman path-integral-like form

$$\begin{aligned}\langle q_N|\hat{U}^{(N)}|q_0\rangle &= \int_{-\infty}^{+\infty}\mathrm{d}x_1 \ldots \int_{-\infty}^{+\infty}\mathrm{d}x_{N-1}\langle x_N|\hat{U}_N|x_{N-1}\rangle \cdots \langle x_1|\hat{U}_1|x_0\rangle \\ &= \left(\sqrt{\frac{m}{2\pi\mathrm{i}\hbar\tau}}\right)^N \int_{-\infty}^{+\infty}\mathrm{d}x_1 \ldots \int_{-\infty}^{+\infty}\mathrm{d}x_{N-1}\exp[(\mathrm{i}/\hbar)S^{(N)}(x_N, \ldots, x_0)]. \end{aligned}\tag{3.4}$$

Here we have used the letters $\{x_n\}$ for the integration variables as they do not coincide with the solutions $\{q_n^{\alpha}\}$ of the classical equation of motion (2.3). Only for initial and end point we have $q_0 = x_0$ and $q_N = x_N$. The action $S^{(N)}(x_N, \ldots, x_0)$ is evaluated along the path $\{x_n\}_{n=0,1,\ldots,N}$.

The "path integral" (3.4) has been approximately calculated by Tabor [7] using the method of stationary phase. We will follow a different but equivalent approach often used in quasi-classical path-integral evaluations [9, 10]. Namely; we will expand the action $S^{(N)}(x_N, \ldots, x_0)$ about the classical path $\{q_n^\alpha\}_{n=0,1,\ldots,N}$. For this we set $x_n =: q_n^\alpha + \xi_n^\alpha$ and $\xi_N^\alpha = 0$, $\xi_0^\alpha = 0$. Expanding up to second order in ξ and neglecting the higher-order terms gives

$$S^{(N)}(x_N, \ldots, x_0) \approx S_{\mathrm{cl}}^\alpha(q_N, q_0) + \tfrac{1}{2} \sum_{n=1}^{N} \left(\frac{m}{\tau}(\xi_n^\alpha - \xi_{n-1}^\alpha)^2 - \tau V_n''(q_n^\alpha)(\xi_n^\alpha)^2 \right). \tag{3.5}$$

The contribution to the propagator from this path then can be put into the form

$$F_N^\alpha \exp[(\mathrm{i}/\hbar) S_{\mathrm{cl}}^\alpha(q_N, q_0)], \tag{3.6}$$

where we have introduced

$$F_N^\alpha := \sqrt{\frac{m}{2\pi \mathrm{i}\hbar\tau}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_1}{\sqrt{\mathrm{i}\pi}} \cdots \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_{N-1}}{\sqrt{\mathrm{i}\pi}} \exp\left[\mathrm{i} \sum_{n=1}^{N} \left((z_n - z_{n-1})^2 - \frac{\tau^2}{m} V_n''(q_n^\alpha)\, z_n^2 \right) \right] \tag{3.7}$$

with $z_n := \xi_n^\alpha \sqrt{(m/2\hbar\tau)}$. The integration is easily performed by diagonalizing the quadratic form in the exponent (note that $z_0 = 0$ and $z_N = 0$) and using the (one-dimensional) Fresnel integral

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{\sqrt{\mathrm{i}\pi}} \mathrm{e}^{\mathrm{i}\lambda z^2} = \frac{1}{\sqrt{|\lambda|}} \begin{cases} 1, & \text{for } \lambda > 0, \\ \mathrm{e}^{-\mathrm{i}\pi/2}, & \text{for } \lambda < 0. \end{cases} \tag{3.8}$$

The result can be written as follows:

$$F_N^\alpha = \sqrt{\frac{m}{2\pi \mathrm{i}\hbar\tau |\det G_N^\alpha|}}\, \mathrm{e}^{-\mathrm{i}\nu_\alpha \pi/2}. \tag{3.9}$$

Here G_N^α is the tridiagonal $(N-1) \times (N-1)$ matrix defined as follows,

$$G_N^\alpha := \begin{pmatrix} d_1^\alpha & -1 & 0 & \cdots & & 0 \\ -1 & d_2^\alpha & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -1 & d_{N-2}^\alpha & -1 \\ 0 & & \cdots & 0 & -1 & d_{N-1}^\alpha \end{pmatrix}, \quad d_n^\alpha := 2 - (\tau^2/m) V_n''(q_n^\alpha), \tag{3.10}$$

and ν_α is a Maslov-like index defined as the number of negative eigenvalues of G_N^α. Expansion along the last row leads to the following recurrence relation for the determinant $D_N^\alpha := \det G_N^\alpha$ of G_N^α

$$D_{n+1}^\alpha = d_n^\alpha D_n^\alpha - D_{n-1}^\alpha, \quad n = 1, 2, \ldots. \tag{3.11}$$

The initial conditions are $D_0^\alpha = 0$ and $D_1^\alpha = 1$. This relation also arises in the time-lattice definition of the path-integral approach to continuous-time quantum systems, where however, the limit $N \to \infty$, $\tau \to 0$ has to be taken such that $N\tau$ remains constant [11].

The complete propagator in the quasi-classical approximation is obtained from (3.6) and (3.9) by summing over all classical paths connecting q_0 and q_N by N time steps τ:

$$\langle q_N|\hat{U}^{(N)}|q_0\rangle \approx \sum_\alpha \sqrt{\frac{m}{2\pi i\hbar\tau|D_N^\alpha|}}\,\exp[(i/\hbar)S_{cl}^\alpha(q_N,q_0) - i\nu_\alpha\pi/2]\,. \tag{3.12}$$

Note that for $N=1$ the index ν_α is zero. However, as N is increasing the matrix G_N^α may develop negative eigenvalues. If we assume that only one of these eigenvalues changes its sign at one instant, the index ν_α is identical to the number of changes in the sign of the determinant D_N^α during its evolution.

4. Discussion of the quasi-classical result

Before studying the properties of the determinant D_N^α we first show the equivalence of our result to that of Tabor [7]. Taking the partial derivative with respect to q_1^α on both sides of the equation of motion (2.3) we find for the expression $\partial q_n/\partial q_1^\alpha$ a recurrence formula which is identical with that of D_n^α and has the same initial conditions. Hence, we obtain for the determinant $D_N^\alpha = \partial q_N/\partial q_1^\alpha$. This is precisely the discrete analogue of what has been found for continuous systems [9, 10]. Tabor's result [7] is established using

$$\frac{\partial^2 S_{cl}^\alpha}{\partial q_N\,\partial q_0} = -\frac{m}{\tau}\frac{\partial}{\partial q_N}(q_1^\alpha - q_0) = -\frac{m}{\tau D_N^\alpha} \tag{4.1}$$

and reads

$$\langle q_N|\hat{U}^{(N)}|q_0\rangle \approx \sum_\alpha \sqrt{\frac{i}{2\pi\hbar}\left|\frac{\partial^2 S_{cl}^\alpha(q_N,q_0)}{\partial q_N\,\partial q_0}\right|}\,\exp[(i/\hbar)S_{cl}^\alpha(q_N,q_0) - i(\nu_\alpha+1)\pi/2]\,, \tag{4.2}$$

which is the well-known Van Vleck formula [5, 10]. Actually, in ref. [7] the extra phases ν_α have been neglected. Also the possibility of having more than one classical path connecting q_0 and q_N has not been taken into account in ref. [7]. Indeed, it is expected [12] that the number of classical irregular paths, similar to that for unstable periodic orbits [4, 5], typically increases exponentially with N. This is one of the fundamental problems in the application of quasi-classical formulas like (3.12) and (4.2).

The advantage of the representation (3.12) over (4.2) is in the simple recurrence relation (3.11) of the determinant which allows for an useful interpretation. Interpreting $Q_n := lD_n^\alpha$ (with α fixed and some arbitrary length scale $l>0$) as a position at time $n\tau$ on the Euclidean line $\mathbb{R}$ and introducing the corresponding momentum $P_n := (Q_{n+1}-Q_n)m/\tau$, the recurrence relation (3.11) can be put into the form

$$Q_n = Q_{n-1} + (\tau/m)P_{n-1}, \qquad P_n = P_{n-1} - \tau V_n''(q_n^\alpha)Q_n. \tag{4.3}$$

This map is generated by a classical system with a quadratic kicking potential, that is, by a Hamiltonian of the form (2.1) with perturbation $\sum_{n=-\infty}^{+\infty}\frac{1}{2}Q^2V_n''(q_n^\alpha)\,\delta(n-t/\tau)$. The initial conditions are $Q_0=0$ and $P_0 = lm/\tau$. The map (4.3) can be used to study the quasi-classical behaviour of rather general systems. For example, for an irregular path the set $\{q_n^\alpha\}$ can be viewed as a sequence of pseudo-random variables and, consequently, (4.3) as the motion of a classical system subjected to a quadratic kicking potential with a random time dependence.

4.1. Analytical investigations

Above we have found that the determinant which appears in the denominator of the propagator is given by $D_N^\alpha = (\partial q_N/\partial q_1^\alpha)|_{q_0\ \text{fixed}}$. In other words, it describes the stability of the classical path $\{q_n^\alpha\}_{n=0,1,\dots,N}$ under a small initial perturbation δq_1^α, that is, $\delta q_N = D_N^\alpha\, \delta q_1^\alpha$. Therefore, we can define a Lyapunov exponent for the path with number α by

$$\lambda_\alpha := \lim_{N\to\infty} \frac{1}{N} \ln\left|\frac{\delta q_N}{\delta q_1^\alpha}\right| = \lim_{N\to\infty} \frac{1}{N} \ln|D_N^\alpha|. \tag{4.4}$$

A path evolving in a regular region of the phase space is stable under the perturbation δq_1^α and we may expect, e.g., $D_N^\alpha \sim \sin\varphi_\alpha N$. On the contrary, an unstable path, that is, a classical path evolving in an irregular region of phase space, deviates exponentially under a small initial perturbation. Hence, for such a path we have $|D_N^\alpha| \sim \exp(\lambda_\alpha N)$ with $\lambda_\alpha > 0$.

These arguments can be made more rigorous by studying the behaviour of the determinant D_N^α in more detail. For this let us put $A_2^\alpha := d_1^\alpha$ and $A_n^\alpha := D_n^\alpha / D_{n-1}^\alpha$ for $n \geq 3$. Then the recurrence relation (3.11) reads

$$A_{n+1}^\alpha = d_n^\alpha - 1/A_n^\alpha, \quad n = 2, 3, \dots . \tag{4.5}$$

Its solution can be expressed in terms of a "reversed continued fraction":

$$A_N^\alpha = d_{N-1}^\alpha - \cfrac{1}{d_{N-2}^\alpha - \cfrac{1}{d_{N-3}^\alpha - \cfrac{1}{\ddots \cfrac{1}{d_2^\alpha - \cfrac{1}{d_1^\alpha}}}}} . \tag{4.6}$$

The determinant is given by the product $D_N^\alpha = \prod_{n=2}^N A_n^\alpha$. Obviously, for large N this product is very sensitive to fluctuations of the magnitude of the A_n^α's about the value 1. If the $|A_n^\alpha|$'s are systematically above the value 1, the determinant increases exponentially with N. Whereas, for more or less symmetric fluctuations about 1 we expect the determinant to be of order unity or to increase at most like some power of N.

This discussion suggests to split the sum in (3.12) into two parts. We classify the paths through their Lyapunov exponents λ_α as follows:

Unstable paths: For these we have $\lambda_\alpha > 0$, that is

$$|D_N^\alpha| \sim \exp(\lambda_\alpha N), \quad N \text{ large}. \tag{4.7}$$

Stable paths: Here we have $\lambda_\alpha = 0$, that is

$$|D_N^\alpha| \sim N^M, \quad M < +\infty, \quad N \text{ large}. \tag{4.8}$$

The sum in the quasi-classical approximation (3.12) should be divided into two parts, that is, $\Sigma_\alpha(\ ) = \Sigma_{\substack{\text{stable}\\ \alpha\text{'s}}}(\ ) + \Sigma_{\substack{\text{unstable}\\ \alpha\text{'s}}}(\ )$. In applying the resulting formula to particular systems one has to find all stable and unstable paths connecting the initial point q_0 and the final point q_N by N time steps τ. This problem is in general not solvable, even numerically. However, there are indirect applications of (3.12) which are very promising [8] (see also our remarks in section 6). The calculation of the classical action, the determinant and the Maslov-like index is very simple. The action is obtained via the sum (2.5), the determinant is found by iterating (3.11) and for the index ν_α one has to count the changes in the sign of D_n^α as n runs from 1 to N.

What are the properties of the propagator in the limit of large N? As for the unstable paths the amplitude in (3.12) is decreasing like $e^{-\lambda_\alpha N/2}$, it is tempting to say that for large N the stable paths are dominating. However, in recalling our remarks made below eq. (4.2), the number of unstable classical paths contributing is also expected to increase exponentially like $e^{\lambda N}$ where λ is a suitable average over the Lyapunov exponents $\{\lambda_\alpha\}$ and is related to the metric entropy [4, 5]. Therefore, the contributions of the unstable paths cannot be neglected in the long-time limit.

How does the determinant and the associated index ν_α typically look like? To answer this question approximately, we replace all the curvatures $\{V_n''(q_n^\alpha)\}$ appearing in the recurrence relation (3.11) by an appropriate (e.g. time) average $\overline{V_n''(q_n^\alpha)} =: m\omega_\alpha^2$. Such an approximation may be called *harmonic* for obvious reasons (see also appendix A). Within this approximation $d_n^\alpha = 2 - \omega_\alpha^2\tau^2$ is independent of n and the recurrence relations (3.11) and (4.5) can be solved exactly. Three cases have to be considered:

(a) $0 \le \omega_\alpha^2\tau^2 \le 4$: We may call such a path *elliptic*. The solutions read

$$A_N^\alpha = \frac{\sin N\varphi_\alpha}{\sin(N-1)\varphi_\alpha}, \qquad D_N^\alpha = \frac{\sin N\varphi_\alpha}{\sin\varphi_\alpha}, \tag{4.9}$$

where we have set $\sin(\frac{1}{2}\varphi_\alpha) := \frac{1}{2}|\omega_\alpha\tau|$, $\varphi_\alpha \in [0, \pi]$. The index ν_α is the integer part of $(N-1)\varphi_\alpha/\pi$. The elliptic paths exist in an area where the curvature of the kicking potential $V_n(q)$ is positive and small.

(b) $\omega_\alpha^2\tau^2 < 0$: Paths belonging to this class we will call *hyperbolic*. Here the solutions read

$$A_N^\alpha = \frac{\cosh N\tilde{\lambda}_\alpha}{\cosh(N-1)\tilde{\lambda}_\alpha}, \qquad D_N^\alpha = \frac{\cosh N\tilde{\lambda}_\alpha}{\cosh\tilde{\lambda}_\alpha}. \tag{4.10}$$

Above we have set $\sinh(\frac{1}{2}\tilde{\lambda}_\alpha) := \frac{1}{2}|\omega_\alpha\tau|$, $\tilde{\lambda}_\alpha > 0$. For the index we obviously have $\nu_\alpha = 0$. The hyperbolic paths live in a region where $V_n(q)$ has negative curvature.

(c) $\omega_\alpha^2\tau^2 > 4$: These paths will be called *inverse hyperbolic* (or *hyperbolic with reflection*). Their solutions are

$$A_N^\alpha = -\frac{\sinh N\tilde{\lambda}_\alpha}{\sinh(N-1)\tilde{\lambda}_\alpha}, \qquad D_N^\alpha = (-1)^{N-1}\frac{\sinh N\tilde{\lambda}_\alpha}{\sinh\tilde{\lambda}_\alpha}, \tag{4.11}$$

with $\cosh(\frac{1}{2}\tilde{\lambda}_\alpha) := \frac{1}{2}|\omega_\alpha \tau|$, $\tilde{\lambda}_\alpha > 0$. The Maslov-like index is $\nu_\alpha = N - 1$. The inverse hyperbolic paths are located in a region where the curvature of $V_n(q)$ is positive and large.

Obviously, the contributions of the hyperbolic paths belong to that of unstable paths. Here the N-dependent parameter $\tilde{\lambda}_\alpha$ may serve for large N as an approximation to the Lyapunov exponent (4.4). The determinant for elliptic paths, which are stable, is expressed by the rotation angle φ_α. Here the long-time behaviour is that we have conjectured in (4.8) with $M = 0$ for $\varphi \in (0, \pi)$ and $M = 1$ for $\varphi = 0$ or π.

We would also like to emphasize the similarity of these formulas to the residues of Greene [13] for stable and unstable periodic orbits. It is this analogy which has suggested the above terminology.

There naturally arises the question whether the harmonic approximation we have made above is a good one or not. For this we will consider the particular example of a quartic kicking potential.

4.2. Numerical results for a quartic kicking potential

In this section we will consider the case of a quartic kicking potential

$$V_n(q) = q^4 \tag{4.12}$$

as a test for the harmonic approximation. Throughout this section we will use the units $m = \tau = 1$. The corresponding dynamical system has previously been studied by Berry et al. [2] in terms of the so-called Wigner propagator. Here we consider directly the quasi-classical approximation of the propagator in the $\hat{q}$-representation by looking at the classical motion. A phase-space portrait of this motion is shown in fig. 1. Obviously, there are only elliptic and inverse hyperbolic paths. We will first consider the elliptic case.

For a classical path to be elliptic it has to be more or less confined to the q-strip in phase space characterized by $V_n''(q) = 12q^2 < 4$, that is $|q| < 1/\sqrt{3} \approx 0.577$. Indeed, from the phase-space portrait shown in fig. 1 we see that all regular paths are inside this strip.

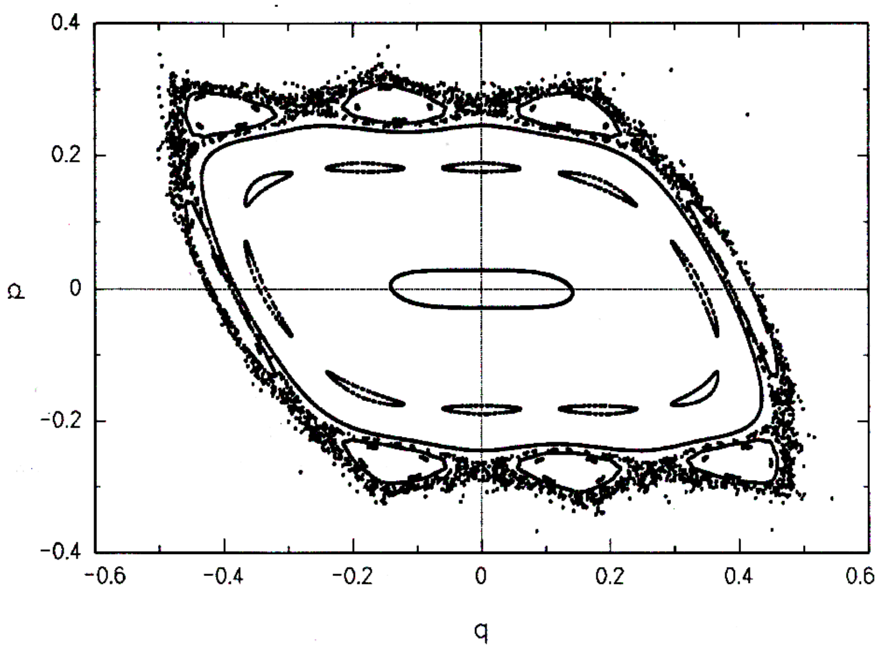

Fig. 1. The phase-space diagram for a quartic kicking potential.

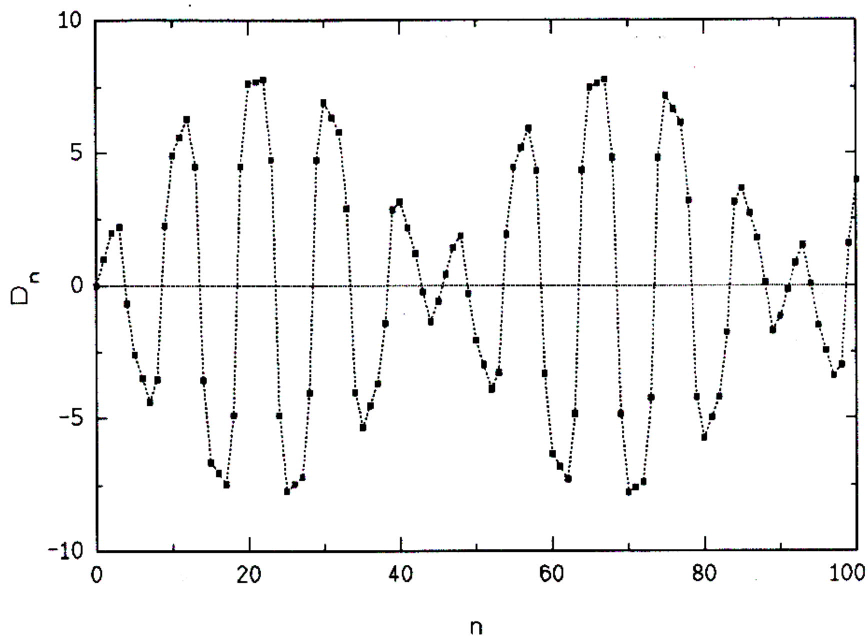

Fig. 2. The determinant D_n for the period-8-orbit. The lines are drawn to guide the eye.

There is only one elliptic path where the above harmonic approximation is actually exact. This is the fixed point at the origin $(p, q) = (0, 0)$ for which we expect $D_N = N$ as follows from (4.9) with $\varphi = 0$. This linear increase can indeed be observed. Elliptic paths which are close to the fixed point also start out linearly. However, after some kicks (this number depends on the distance of the path to the fixed point), the determinant starts with regular oscillations as predicted by (4.9) but in addition the amplitude is linearly increasing.

There is also another class of regular paths where we can expect our approximation to be good. These are the periodic orbits. Here the curvature of the potential seen by the paths is no longer constant. It oscillates periodically with the period of the orbit. These additional oscillations should also be seen in the determinant (4.9) as an extra modulation. Again, this can be observed. In fig. 2 we show the behaviour of the determinant for the period-8-orbit. Note that the amplitude remains bounded. This is an essential property of all stable periodic orbits and gives rise to singularities in the sum over periodic orbits [9]. For elliptic paths staying near a stable periodic orbit the determinant shows the same oscillations but the amplitude is in addition linearly increasing.

As an inverse hyperbolic path we consider the irregular path starting in $(p_0, q_0) = (0.310, 0.000)$. It leaves the region of phase space displayed in fig. 1 after about 1300 iterations. We observe the oscillations in the sign of the determinant as predicted by (4.11). In fig. 3 we also see the exponential increase in the modulus as expected from (4.7). The straight line indicates a least-square fit to the long-time behaviour (4.7) from which one could read off the approximate Lyapunov exponent $\tilde{\lambda}_\alpha$.

4.3. *Implications for spectral properties*

The propagator does also provide information about the quasi-energy spectrum and the corresponding eigenfunctions. For studying these quantities we assume in this section that the kicking potential does not explicitly depend on time, that is, $V_n(q) = V_1(q)$ which gives $\hat{U}^{(N)} = (\hat{U}_1)^N$. Furthermore, as $\hat{U}_1$ is

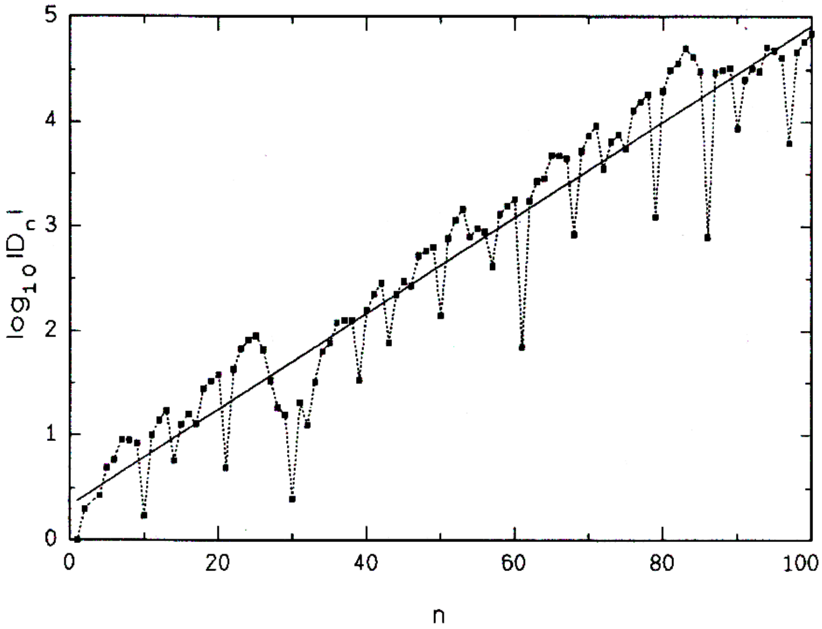

Fig. 3. $\log_{10}|D_n|$ for the irregular path starting in $(p_0, q_0) = (0.310, 0.000)$. The straight line is a fit to the long-time behaviour (4.7). Its slope determines the approximate Lyapunov exponent $\bar{\lambda}_\alpha$.

unitary we can set $\hat{U}_1 = \exp\{-\mathrm{i}\tau\hat{E}/\hbar\}$ where the self-adjoint operator $\hat{E}$ is the so-called quasi-energy operator.

Let us introduce the "Fourier transform" of the causal propagator,

$$\hat{G}(E) := \sum_{N=1}^{\infty} \hat{U}^{(N)} \mathrm{e}^{(\mathrm{i}/\hbar)N\tau E} = \frac{1}{\exp\left[\mathrm{i}\tau(\hat{E}-E)/\hbar\right] - 1}, \quad \operatorname{Im} E > 0. \tag{4.13}$$

From the spectral resolution of $\hat{E}$, that is, $\hat{E} = \sum_\mu E_\mu |\psi_\mu\rangle\langle\psi_\mu|$ where the set $\{E_\mu\}$ forms the quasi-energy spectrum and the set $\{|\psi_\mu\rangle\}$ the corresponding (generalized) eigenvectors, we find for the diagonal elements of $\hat{G}(E)$ in the $\hat{q}$-representation

$$\langle q|\hat{G}(E)|q\rangle = \sum_\mu \frac{|\langle q|\psi_\mu\rangle|^2}{\exp\left[\mathrm{i}\tau(E_\mu - E)/\hbar\right] - 1}. \tag{4.14}$$

For a continuous part of the spectrum the sum has to be replaced by an integral. The poles of $\langle q|\hat{G}(E)|q\rangle$ provide us information about the discrete part of the spectrum, whereas, their weights are related to the corresponding eigenfunctions.

In the quasi-classical approximation (3.12) we can represent the above diagonal element as follows

$$\langle q|\hat{G}(E)|q\rangle \approx 2\sqrt{\frac{m}{2\pi\mathrm{i}\hbar\tau}} \sum_{N=1}^{\infty} \sum_{\alpha(N)} \frac{1}{\sqrt{|D_n^\alpha|}} \exp\left[\mathrm{i}(S_{\mathrm{cl}}^\alpha(q,q) + N\tau E)/\hbar - \mathrm{i}\nu_\alpha\pi/2\right]. \tag{4.15}$$

Here the sums cover all closed, not necessarily periodic, orbits starting in q and coming back to q after an *arbitrary* number of kicks $N \geq 1$. The extra factor of two appearing in front of the sums is due to the time-reversal invariance of systems with $V_n(q) = V_1(q)$. If q belongs to a periodic orbit there is an infinite number of terms in the sum of (4.15) as each full cycle is counted separately. For a stable periodic orbit the result (4.9) of the harmonic approximation indicates that the N-sum diverges which, in turn, suggests a discrete part in the quasi-energy spectrum. For an unstable periodic orbit (hyperbolic or inverse hyperbolic), the sum over N converges and gives a finite enhanced contribution to the wave functions. This may account for the observed scaring of wave functions at and near hyperbolic periodic orbits [14, 15]. To be more explicit, the contribution to $\langle q|\hat{G}(E)|q\rangle$ from an elliptic, a hyperbolic and an inverse hyperbolic periodic orbit with period $p \in \mathbb{N}$ reads within the harmonic approximation

$$\sqrt{\frac{2m \sin \varphi_\alpha}{\mathrm{i}\pi\hbar\tau}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{|\sin kp\varphi_\alpha|}} \exp\{\mathrm{i}k[(S_{\mathrm{cl}}^\alpha(q,q) + p\tau E)/\hbar - \nu_\alpha\pi/2]\},$$

$$\sqrt{\frac{2m \cosh \tilde{\lambda}_\alpha}{\mathrm{i}\pi\hbar\tau}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{\cosh kp\tilde{\lambda}_\alpha}} \exp\{\mathrm{i}k[S_{\mathrm{cl}}^\alpha(q,q) + p\tau E]/\hbar\},$$

$$\sqrt{\frac{2m \sinh \tilde{\lambda}_\alpha}{\mathrm{i}\pi\hbar\tau}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{\sinh kp\tilde{\lambda}_\alpha}} \exp\{\mathrm{i}k[(S_{\mathrm{cl}}^\alpha(q,q) + p\tau E)/\hbar - (p-1)\pi/2]\}, \tag{4.16}$$

respectively, where ν_α is the integer part of $(p-1)\varphi_\alpha/\pi$.

5. Higher-order corrections

In order to find the next-order corrections in $\hbar$ to the quasi-classical propagator, we have to take two more terms in the expansion (3.6) into account. These are

$$\delta^3 S^\alpha := -\frac{\tau}{6} \sum_{n=1}^{N-1} V_n'''(q_n^\alpha)\,(\xi_n^\alpha)^3, \tag{5.1}$$

$$\delta^4 S^\alpha := -\frac{\tau}{24} \sum_{n=1}^{N-1} V_n^{(4)}(q_n^\alpha)\,(\xi_n^\alpha)^4. \tag{5.2}$$

As the term $\delta^3 S^\alpha$ does not contribute in the first order of the perturbation theory, both corrections are of the same order in $\hbar$. The explicit calculation gives for the corrected amplitude (3.9):

$$F_N^\alpha = \sqrt{\frac{m}{2\pi\mathrm{i}\hbar\tau|\det G_N^\alpha|}}\, \mathrm{e}^{-\mathrm{i}\nu_\alpha\pi/2} \Bigg(1 + \frac{\mathrm{i}\hbar\tau^3}{8m^2} \sum_{n=1}^{N-1} V_n^{(4)}(q_n^\alpha)\,(G^{-1})_{nn}^2 + \frac{\mathrm{i}\hbar\tau^5}{24m^3} \sum_{n,l=1}^{N-1} V_n'''(q_n^\alpha)\,V_l'''(q_l^\alpha)\left[2(G^{-1})_{nl}^3 + 3(G^{-1})_{nn}(G^{-1})_{nl}(G^{-1})_{ll}\right]\Bigg). \tag{5.3}$$

In the above $(G^{-1})_{nl}$ denotes the matrix elements of the inverse matrix in (3.10). Due to the tridiagonal form of G_N^α, the diagonal elements of G^{-1} are given by

$$(G^{-1})_{nn} = \frac{1}{\det G_N^\alpha} \begin{vmatrix} d_1^\alpha & -1 & 0 & \cdots & & 0 \\ -1 & d_2^\alpha & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -1 & d_{n-2}^\alpha & -1 \\ 0 & & \cdots & 0 & -1 & d_{n-1}^\alpha \end{vmatrix}$$
$$\times \begin{vmatrix} d_{n+1}^\alpha & -1 & 0 & \cdots & & 0 \\ -1 & d_{n+2}^\alpha & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -1 & d_{N-2}^\alpha & -1 \\ 0 & & \cdots & 0 & -1 & d_{N-1}^\alpha \end{vmatrix}. \tag{5.4}$$

Near the transition from an elliptic to a hyperbolic path, that is $\varphi \to 0$ or π, we have found in section 4.1 that $|\det G_N^\alpha| \sim N$. For the above diagonal elements this leads to $|(G^{-1})_{nn}| \sim n(N-n)/N \le \frac{1}{4}N$. Hence, the correction due to the $\delta^4 S^\alpha$ term may be estimated by

$$|(i/\hbar)\delta^4 S^\alpha| \le \frac{\hbar\tau^3 N^3}{8(4m)^2} \sup_n |V_n^{(4)}(q_n^\alpha)|. \tag{5.5}$$

Assuming that the off-diagonal elements of G^{-1} obey the same upper bound, that is $|(G^{-1})_{nl}| \le \frac{1}{4}N$, we can also estimate the $\delta^3 S^\alpha$ contribution:

$$\tfrac{1}{2}|(i/\hbar)\delta^3 S^\alpha|^2 \le \frac{5\hbar\tau^5 N^5}{24(4m)^3} \sup_n |V_n'''(q_n^\alpha)|^2. \tag{5.6}$$

If the dominant term of the higher-order corrections in $\hbar$ is due to $\delta^4 S^\alpha$, the upper bound in (5.5) has to be much smaller than unity for the validity of the quasi-classical approximation. This leads to the following upper bound for the number of iterations:

$$N \ll \text{constant}/\hbar^{1/3}. \tag{5.7}$$

On the other hand, if (5.6) is dominating we find for the upper bound of N:

$$N \ll \text{constant}/\hbar^{1/5}. \tag{5.8}$$

A similar scaling occurring just before the transition from regular to irregular motion has also been found by Fishman et al. [16]. Beyond this transition, when the contribution of the unstable paths dominates in (3.12), a different bound

$$N \ll \text{constant} \times \ln(1/\hbar) \tag{5.9}$$

can be obtained (see also refs. [3, 6]).

6. Concluding remarks

In this article we have derived and discussed the quasi-classical propagator of area-preserving maps. Our result obtained by expansion about classical paths is certainly the same as that obtained from the method of stationary phase [7]. However, the present result (3.12) expressed in terms of the determinant D_N^α provides more insight into the quasi-classical behaviour. The contributions to the quasi-classical propagator are naturally split into two parts due to stable and unstable classical paths. These contributions have been estimated by making a harmonic approximation for their amplitude. Comparison with the exact propagator for a quadratic kicking potential which can be found in appendix A suggests that the stable paths are responsible for a discrete part in the quasi-energy spectrum of the system. We have also analyzed the Maslov-like indices for the paths. For hyperbolic paths they are indeed vanishing as claimed by Tabor [7]. However, for inverse hyperbolic and elliptic paths these indices cannot be neglected. It is well known that these phases effect the energy spectrum in the quasi-classical approximation [9, 10].

A rough estimate for the validity of the quasi-classical approximation has also been given. Near the stochastic transition where all regular paths become irregular we have found a scaling behaviour for the maximum number of kicks for which the present approximation is still valid. It is interesting to note that this behaviour is similar to the one which has been found by a renormalization-group approach [16].

Another advantage of the present approach is in the simple recurrence relations (3.11) and (4.3) which make a numerical investigation of classical chaotic systems at the quasi-classical level very easy. For example, the so-called correlation function, i.e. the overlap of a state vector $|\Psi\rangle$ with a state vector $|\Phi\rangle$ after N kicks,

$$\langle\Psi|\hat{U}^{(N)}|\Phi\rangle = \int_{-\infty}^{+\infty} \mathrm{d}q_N \int_{-\infty}^{+\infty} \mathrm{d}q_0 \langle\Psi|q_N\rangle\langle q_N|\hat{U}^{(N)}|q_0\rangle\langle q_0|\Phi\rangle, \tag{6.1}$$

may be considered in the present approximation. A numerical evaluation of this integral for well-localized initial and final wave packets is easily performed by iterating the classical map (2.2). The classical action given in (2.5), the determinant obtained through (3.11) or (4.3) and the Maslov-like index are by-products of this iteration. For a recent study of these correlation functions for a continuous-time autonomous system see ref. [8].

We would also like to mention that the present result is only valid if the configuration space underlying (2.1) is the full Euclidean line $\mathbb{R}$. Systems with cyclic boundary conditions such as the standard map corresponding to the kicked rotator [1] have the unit circle $\mathscr{S}^1 := \{q | q \in [0, 2\pi)\}$ as its configuration space. For such maps the correct position representation of the one-kick propagator reads

$$\langle x|\hat{U}_n|x'\rangle = \sqrt{\frac{m}{2\pi i\hbar\tau}} \sum_{k=-\infty}^{+\infty} \exp[(i/\hbar)S_n(x, x' + 2\pi k)]. \tag{6.2}$$

Therefore, for the standard map, as discussed by Tabor [7], the quasi-classical formulas (3.12) and (4.2) need appropriate modifications.

Finally, we point out that the present approach can be generalized to the kicked harmonic oscillator

$$H(t) = \frac{p^2}{2m} + \tfrac{1}{2}m\Omega_0^2 q^2 + \sum_{n=-\infty}^{+\infty} V_n(q)\,\delta(n - t/\tau), \quad (p, q) \in \mathbb{R}^2. \tag{6.3}$$

Acknowledgements

One of the authors (G.J.) would like to thank M.V. Berry and M.C. Gutzwiller for clarifying discussions.

Appendix A. An exact result for a quadratic kicking potential

For a quadratic (time-independent) kicking potential

$$V_n(q) = \tfrac{1}{2}m\omega^2 q^2, \tag{A.1}$$

the truncated quasi-classical expansion in (3.5) is exact. In this case the harmonic approximation performed in section 4.1 is also exact. For the elliptic case $0 \le \omega^2\tau^2 \le 4$, $\sin\frac{1}{2}\varphi := \frac{1}{2}|\omega\tau|$, the exact quantum propagator reads

$$\langle q_N|\hat{U}^{(N)}|q_0\rangle = \sqrt{\frac{m\sin\varphi}{2\pi i\hbar\tau|\sin N\varphi|}}\,\exp[(i/\hbar)S_{cl}(q_N,q_0) - i\nu\pi/2]. \tag{A.2}$$

It is a well-known fact that the Van Vleck formula (4.2) gives the correct propagator for all quadratic Lagrangians which is essentially a harmonic oscillator with time-dependent mass $m(t)$ and time-dependent frequency $\omega(t)$ [17]. Here we have the special case $m(t) = m$ and $\omega^2(t) = \omega^2\Sigma_n\delta(n - t/\tau)$. For $0 \le \omega^2\tau^2 \le 4$ the classical motion is stable and all paths are elliptic orbits. For $\omega^2\tau^2 > 4$ all paths become unstable (inverse hyperbolic) and there exist no bound states. The same is valid for the quadratic repeller, that is $\omega^2 < 0$ (hyperbolic paths).

The classical action can also be found in closed form,

$$S_{cl}(q_N,q_0) = \frac{m\sin\varphi}{2\tau\sin N\varphi}\left[(q_N^2 + q_0^2)\cos N\varphi - 2q_N q_0\right] + \tfrac{1}{4}m\omega^2\tau(q_0^2 - q_N^2), \tag{A.3}$$

which is that of an ordinary (that is time-independent) harmonic oscillator with frequency $\Omega := (2/\tau)\arcsin\frac{1}{2}\omega\tau$ and mass $M := (\frac{1}{2}m\sin\varphi)/(\arcsin\frac{1}{2}\omega\tau)$. Note that $\sin\varphi = \tau\omega(1 - \frac{1}{4}\tau^2\omega^2)^{1/2}$. This identification, which has already been realized by Berry et al. [2], leads to the interesting relation

$$\left[\exp\left(-\frac{i}{\hbar}\frac{m\omega^2}{2}\hat{q}^2\tau\right)\exp\left(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\tau\right)\right]^N$$
$$= \exp\left(-\frac{i}{\hbar}\frac{m\omega^2}{4}\hat{q}^2\tau\right)\exp\left[-\frac{i}{\hbar}\left(\frac{\hat{p}^2}{2M} + \frac{M\Omega^2}{2}\hat{q}^2\right)N\tau\right]\exp\left(\frac{i}{\hbar}\frac{m\omega^2}{4}\hat{q}^2\tau\right). \tag{A.4}$$

In other words, for a free system periodically kicked with a quadratic potential the product (3.1) can indeed be expressed in closed form. For an operator approach to this product formula see ref. [18].

Appendix B. The quasi-classical trace formula

Finally, we would like to point out that in our approach the trace of the propagator is obtained in a straightforward way. The trace which is defined by

$$\operatorname{Tr} \hat{U}^{(N)} := \int_{-\infty}^{+\infty} \mathrm{d}x \, \langle x | \hat{U}^{(N)} | x \rangle = \int_{-\infty}^{+\infty} \mathrm{d}x_1 \ldots \int_{-\infty}^{+\infty} \mathrm{d}x_N \langle x_N | \hat{U}_N | x_{N-1} \rangle \cdots \langle x_1 | \hat{U}_1 | x_N \rangle \tag{B.1}$$

may be calculated quasi-classically by expanding about classical closed orbits of length N to second order. Such paths are characterized by the condition $q_N(q_0, q_1) = q_0$. Note that $q_0 = q_N$ is no longer fixed. By considering the integration over this variable (using the method of stationary phase) we get the condition $p_0 = p_N$. Therefore, not all closed paths do contribute in lowest order to the trace (B.1). Only the periodic orbits having a period being an integer fraction of N have to be considered to this order in $\hbar$. Let the number β enumerate all these periodic orbits which are given by the set $\{q_n^\beta\}_{n=1,\ldots,N}$. Actually, only a fraction (identical to the period) of this set is needed to describe this orbit.

The integration above may now be performed in analogy to section 3. Accordingly, we expand about a periodic orbit with number β to second order and perform the integration. The quasi-classical approximation for the trace (B.1) reads

$$\operatorname{Tr} \hat{U}^{(N)} \approx \sum_{\beta} \mathscr{F}_N^\beta \exp\left[(\mathrm{i}/\hbar) S_{\mathrm{cl}}^\beta(N)\right], \tag{B.2}$$

where $S_{\mathrm{cl}}^\beta(N) := S_{\mathrm{cl}}(q_N^\beta, q_0^\beta)$ is the classical action corresponding to the periodic orbit with number β and

$$\mathscr{F}_N^\beta := \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_1}{\sqrt{\mathrm{i}\pi}} \cdots \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_N}{\sqrt{\mathrm{i}\pi}} \exp\left[\mathrm{i} \sum_{n=1}^{N} \left((z_n - z_{n-1})^2 - \frac{\tau^2}{m} V_n''(q_n^\beta)\, z_n^2 \right)\right]. \tag{B.3}$$

Note that $z_0 := z_N$ does not vanish in the present case, but is also an integration variable. The integration is done using again (3.8) and yields

$$\mathscr{F}_N^\beta = \frac{1}{\sqrt{|\det \mathscr{G}_N^\beta|}} \, \mathrm{e}^{-\mathrm{i}\nu_\beta \pi/2}, \tag{B.4}$$

where

$$\mathscr{G}_N^\beta := \begin{pmatrix} d_1^\beta & -1 & 0 & \cdots & 0 & -1 \\ -1 & d_2^\beta & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -1 & d_{N-1}^\beta & -1 \\ -1 & 0 & \cdots & 0 & -1 & d_N^\beta \end{pmatrix}, \quad d_n^\beta = 2 - \frac{\tau^2}{m} V_n''(q_n^\beta), \tag{B.5}$$

is a generalization of the stability matrix introduced by Bountis and Helleman [19] and ν_β is the number of negative eigenvalues of the matrix $\mathscr{G}_N^\beta$. This $N \times N$ matrix is related to the residue R_N^β of Greene [13] by $R_N^\beta = -\frac{1}{4}\det \mathscr{G}_N^\beta$. Periodic orbits are usually classified into elliptic, hyperbolic and inverse hyperbolic type. Their residues, see refs. [13, 19], are very similar to the expressions given in section 4.1.

The trace of the propagator reads

$$\mathrm{Tr}\, \hat{U}^{(N)} \approx \sum_\beta \frac{1}{2\sqrt{|R_N^\beta|}} \exp\left[(\mathrm{i}/\hbar) S_{\mathrm{cl}}^\beta(N) - \mathrm{i}\nu_\beta \pi/2\right]. \tag{B.6}$$

While this result is of the same form as the one obtained by Tabor [7] we note two differences. First, in ref. [7] the trace formula contains all periodic orbits whereas only those with a period being an integer fraction of N should be considered. It is the trace of the operator $\hat{G}(E)$ defined in (4.13) which contains *all* periodic orbits. Secondly, the Maslov-like phases have also been neglected by Tabor [7]. Finally, we note that we allow for an explicit time dependence in the kicking potential $V_n(q)$. Therefore, the time-inversion invariance of periodic orbits is broken in general.

References

[1] G. Casati, B.V. Chirikov, F.M. Izraelev and J. Ford, in: Stochastic Behaviour in Classical and Quantum Hamiltonian Systems, Springer Lecture Notes in Physics Vol. 93, eds. G. Casati and J. Ford (Springer, Berlin, 1979) p. 334.

[2] M.V. Berry, N.L. Balazs, M. Tabor and A. Voros, Ann. Phys. (NY) 122 (1979) 26.

[3] G.M. Zaslavsky, Phys. Rep. 80 (1981) 157;
B. Eckhardt, Phys. Rep. 163 (1988) 205.

[4] A.M. Ozorio de Almeida, Hamiltonian Systems: Chaos and Quantization (Cambridge Univ. Press, Cambridge, 1988).

[5] M.C. Gutzwiller, Chaos in Classical and Quantum Mechanics (Springer, New York, 1990).

[6] F. Haake, Quantum Signatures of Chaos (Springer, Berlin, 1991).

[7] M. Tabor, Physica D 6 (1983) 195.

[8] S. Tomsovic and E.J. Heller, Phys. Rev. Lett. 36 (1991) 664.

[9] M.C. Gutzwiller, J. Math. Phys. 12 (1971) 343.

[10] L.S. Schulman, Techniques and Applications of Path Integration (Wiley, New York, 1981).

[11] I.M. Gel'fand and A.M. Yaglom, J. Math. Phys. 1 (1960) 48;
For a recent discussion on the Gel'fand–Yaglom formula see also: H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics (World Scientific, Singapore, 1990).

[12] M.V. Berry, private communication.

[13] J.M. Greene, J. Math. Phys. 20 (1979) 1183.

[14] E.B. Bogomolny, Physica D 31 (1988) 169.

[15] G. Radons and R.E. Prange, in: Quantum Chaos, eds. H.A. Cerdeira, M.C. Gutzwiller, R. Ramaswamy and G. Casati (World Scientific, Singapore, 1991). (Trieste, Italy, 1990) to appear.

[16] S. Fishman, D.R. Grempel and R.E. Prange, Phys. Rev. A 36 (1987) 289.

[17] G. Junker and A. Inomata, Phys. Lett. A 110 (1985) 192.

[18] R. Blümel, R. Meir and U. Smilansky, Phys. Lett. A 103 (1984) 353.

[19] T. Bountis and R.H.G. Helleman, J. Math. Phys. 22 (1981) 1867.